Practice Final Exam III

We strongly recommend that you work through this exam under realistic conditions rather than just flipping through the problems and seeing what they look like. Setting aside three hours in a quiet space with your notes and making a good honest effort to solve all the problems is one of the single best things you can do to prepare for this exam. It will give you practice working under time pressure and give you an honest sense of where you stand and what you need to get some more practice with.

This practice final exam is essentially the final exam from Fall 2017, with one or two questions swapped out for questions from previous quarter's final exams. The sorts of questions here are representative of what you might expect to get on the upcoming final exam, though the point balance and distribution of problems might be a bit different.

The exam policies are the same for the midterms – closed-book, closed-computer, limited note (one double-sided sheet of 8.5” × 11” paper decorated however you'd like).

You have three hours to complete this exam. There are 62 total points.

Question	Points	Graders
(1) Mathematical Logic and Set Theory	/ 8	
(2) Functions and Binary Relations	/ 16	
(3) Number Theory and Induction	/ 12	
(4) Regular and Context-Free Languages	/ 10	
(5) **R** and **RE** Languages	/ 10	
	/ 56	

Problem One: Mathematical Logic and Set Theory (8 Points)

(Final Exam, Spring 2015)

i. **(5 Points)** Suppose we have the predicates

- $Person(p)$, which states that p is a person, and
- $Loves(p, q)$, which states that p loves q.

Below are a series of five English statements paired with a statement in first-order logic. For each statement, decide whether the corresponding formula in first-order logic is a correct translation of the English statement and check the appropriate box. There is no penalty for an incorrect guess.

English statement	First-order logic	
Everyone loves themselves.	$\forall p. (Person(p) \to \forall q. (Loves(p, q) \to p = q))$	☐ ***Correct*** ☐ ***Incorrect***
There are two people that everyone loves.	$\forall r. (Person(r) \to \exists p. (Person(p) \land \exists q. (Person(q) \land q \neq p \land Loves(r, p) \land Loves(r, q))))$	☐ ***Correct*** ☐ ***Incorrect***
Love is a transitive relation over the set of people.	$\forall p. (Person(p) \land \forall q. (Person(q) \land \forall r. (Person(r) \land (Loves(p, q) \land Loves(q, r) \to Loves(p, r)))))$	☐ ***Correct*** ☐ ***Incorrect***
No two people love exactly the same set of people.	$\forall p. (Person(p) \to \forall q. (Person(q) \land q \neq p \to \exists r. (Person(r) \land (Loves(p, r) \leftrightarrow \neg Loves(q, r)))))$	☐ ***Correct*** ☐ ***Incorrect***
Someone doesn't love anyone.	$\neg\forall p. (Person(p) \to \exists q. (Person(q) \land Loves(p, q)))$	☐ ***Correct*** ☐ ***Incorrect***

In a mathematical sense, we can think of a committee as a set whose elements are the people on that committee. This lets us talk about different committees in the language of set theory.

ii. **(3 Points)** Let S be the set of all people in the United States, R be the set of all Republicans, and D be the set of all Democrats. Using set theory notation (e.g. $\cup$, $\subseteq$, $\wp$, $\in$, etc.), but ***without*** using set-builder notation and ***without*** using first-order logic, write an expression that represents the set of all possible committees of people from the US that include at least one Democrat and at least one Republican.

    As before, remember that some people may be neither Republicans nor Democrats.

Problem Two: Functions and Binary Relations (16 Points)

(Final Exam, Fall 2017)

On Problem Sets Three and Four you explored different properties of strict orders. Although strict orders come in all sorts of shapes and flavors, there is a single strict order that's, in some sense, the "most fundamental" strict order: the strict subset relation. In this problem, you'll show that every strict order's behavior can be thought of as the behavior of the strict subset relation over some well-chosen collection of sets.

Let R be a strict order over a set A. Consider the function $f : A \to \wp(A)$ defined as follows:

$$f(x) = \{ \ y \in A \ \ | \ \ y = x \ \text{ or } \ yRx \ \}$$

This function f connects the relation R over A to the relation $\subsetneq$ over $\wp(A)$.

i. **(6 Points)** Prove for all $a, b \in A$ that if $f(a) \subsetneq f(b)$, then aRb. As a reminder, the notation $S \subsetneq T$ means that $S \subseteq T$ and that $S \neq T$.

   Feel free to use the space below for scratch work. There's room for your answer to this question on the next page of this exam.

(Extra space for your answer to Problem Two, Part (i), if you need it.)

As a refresher from the previous page, we've let R be a ***strict order*** over a set A and defined the function $f : A \to \wp(A)$ as follows:

$$f(x) = \{\ y \in A \ \ | \ \ y = x \ \text{ or } \ yRx\ \}$$

ii. **(10 Points)** Prove for all $a, b \in A$ that if aRb, then $f(a) \subsetneq f(b)$. Again, the notation $S \subsetneq T$ means that $S \subseteq T$ and that $S \neq T$.

Feel free to use this space for scratch work. There's room to write your answer to this question on the next page of the exam.

(Extra space for your answer to Problem Two, Part (ii), if you need it.)

Problem Three: Number Theory and Induction (12 Points)

(Midterm Exam, Fall 2018)

On Problem Set Four, you explored recurrence relations and number theory. This problem is designed to give you a chance to demonstrate what you've learned in the process.

Let's begin with a refresher on some definitions. First, if x and y are integers, we say x ***divides*** y if there is an integer q such that $y = xq$. Second, if a and b are integers, we say that $a \perp b$ (a and b are ***coprime***) if the only integers that divide both a and b are ± 1.

i. **(5 Points)** Let a and b be arbitrary integers. Prove that if $a \perp b$, then $a+b \perp a$.

(Extra space for your answer to Problem Four, Part (i), if you need it.)

The ***Fibonacci numbers*** are a sequence of natural numbers defined by the following recurrence relation:

$$F_0 = 0$$

$$F_1 = 1$$

$$F_{n+2} = F_{n+1} + F_n.$$

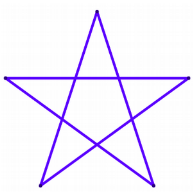
The first few terms of the Fibonacci sequence are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots .$$

This question explores a nifty property of the Fibonacci numbers.

ii. **(7 Points)** Using your result from part (i), prove that $F_n \perp F_{n+1}$ for every $n \in \mathbb{N}$. Remember that $0 \in \mathbb{N}$.

Just for fun: this result, combined with what you learned on Problem Set Five, says that the star $\{F_{n+1} / F_n\}$ is a simple for any natural number n. As you can see, these stars are very pretty!



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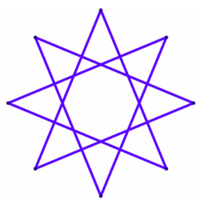

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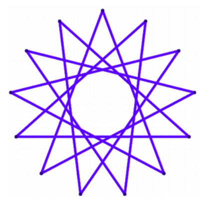

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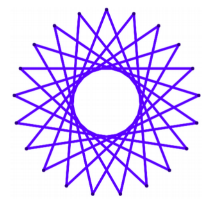

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(Extra space for your answer to Problem Three, Part (ii), if you need it.)

Problem Four: Regular and Context-Free Languages (10 Points)

(Final Exam, Fall 2017)

Let Σ = { `h`, `i`, `m`, `r`, `t` } and consider the following language L_1:

$$L_1 = \{ w \in \Sigma^* \mid w \text{ is a substring of } \texttt{mirth} \}.$$

Recall that a substring is a contiguous range of characters taken out of an original string. For example, `mir` $\in L_1$, `irt` $\in L_1$, $\varepsilon \in L_1$, `t` $\in L_1$, and `mirth` $\in L_1$, but `mrh` $\notin L_1$ (although the letters in `mrh` appear in `mirth`, they're not contiguous), `it` $\notin L_1$ (for the same reason `mrh` is not in L_1), and `mmm` $\notin L_1$ (because there aren't three consecutive `m`'s in `mirth`).

i. **(3 Points)** Design an NFA for L_1. In the space at the bottom of the page, write a brief explanation (at most two sentences) for how your NFA works.

Explanation for this NFA (at most two sentences):

On Problem Set Five, you explored languages involving taking a walk with your dog. The next two parts of this problem concern more of the challenges of pet ownership.

Imagine that you and your dog are taking a walk and you have a leash that's six units long. Unlike before, you and your dog don't move at the same speed. Every time your dog takes a step, your dog moves three units forward, and every time you take a step, you move two units forward.

Let Σ = {y, d}, where y represents you taking a step (which moves you two units forward) and d represents your dog taking a step (which moves your dog three units forward) and consider the following language:

L_2 = { $w \in \Sigma^*$ | w represents a walk where you and your dog end at the same position,
you and your dog are never more than six units apart, and
you never are ahead of your dog. }

For example, the string ddyyy ∈ L_2, as are dydydydyyy, ε, dydyyddyyy, and dydydyydyy. However, the string yddyy ∉ L_2 (since after the first step you end up ahead of your dog), the string ddydyyyyy ∉ L_2 (at the underlined point, your dog is more than six units ahead of you), and the string dy ∉ L_2 (your dog ends up one unit ahead of you).

ii. **(4 Points)** Design a DFA for L_2. At the bottom of the page, write a very brief explanation (at most two sentences) about how your DFA works.

Explanation for this DFA (at most two sentences):

Now, let's imagine that you're taking your dog for a walk but you take off the leash. As before, every time your dog takes a step it moves three units forward, and every time you take a step you move two units forward.

Let Σ = {y, d}, where y represents you taking a step (which moves you two units forward) and d represents your dog taking a step (which moves your dog three units forward) and consider the following language L_3:

L_3 = { *w* ∈ Σ* | *w* represents a walk where you and your dog end at the same position and you never are ahead of your dog. }

This is essentially the same language as L_2, except without the leash restriction. This language is not regular, and in this problem we'd like you to convince us why this is.

iii. **(3 Points)** To save you time, rather than having you write out a full proof that L_3 is not regular, we'd instead like you to answer the following questions:

1. The Myhill-Nerode theorem requires you to choose an infinite set *S* of strings that are pairwise distinguishable relative to L_3. In the space below, write such a set *S*.

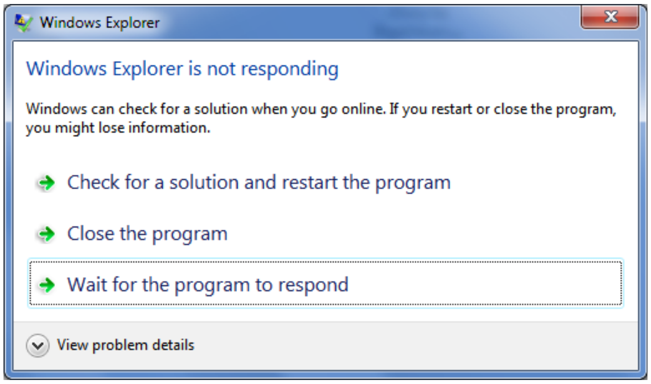
2. Suppose you choose two arbitrary, distinct strings *x* and *z* from the set *S*. What string *w* will you append to *x* and *z* to show that they are distinguishable relative to L_3?

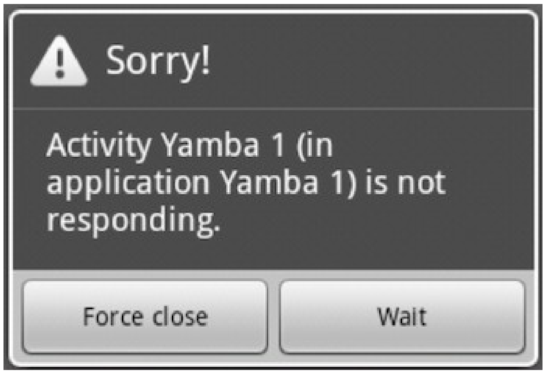
3. Briefly explain why the choice of *w* you found above shows that *x* and *z* are distinguishable relative to L_3.

Problem Five: R and RE Languages (10 Points)

(Final Exam, Fall 2011)

Most operating systems provide some functionality to detect programs that are looping infinitely. Typically, they display a dialog box containing a message like these:



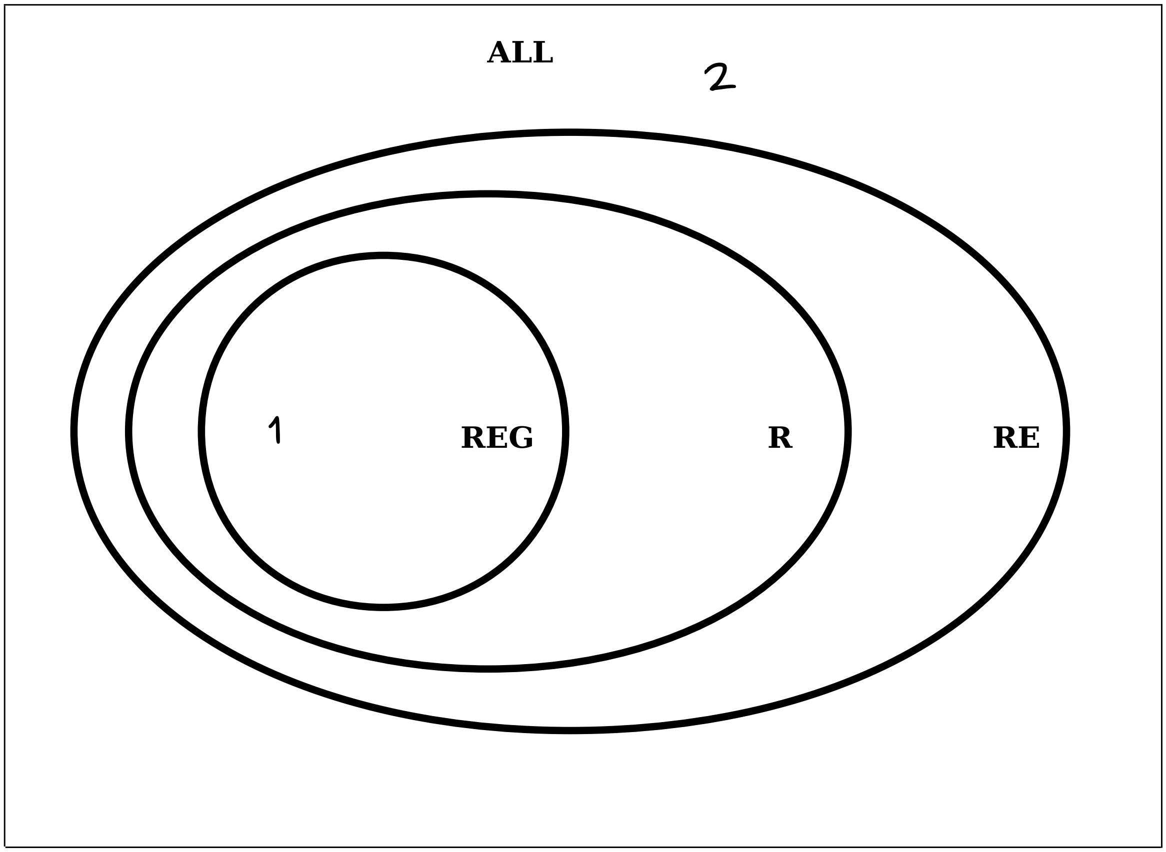


These messages give the user the option to terminate the program or to let the program keep running.

An ideal OS would shut down any program that had gone into an infinite loop, since these programs just waste system resources (processor time, battery power, etc.) that could be better spent by other programs.

i. **(3 Points)** Since it makes more sense for the OS to automatically detect programs that have gone into an infinite loop, why does the OS have to ask the user whether to terminate the program or let it keep running?

ii. **(7 Points)** Below is a Venn diagram showing the overlap of different classes of languages we've studied so far. We have also provided you a list of numbered languages. For each of those languages, draw where in the Venn diagram that language belongs. As an example, we've indicated where Language 1 and Language 2 should go. No proofs or justifications are necessary, and there is no penalty for an incorrect guess.



1. Σ^*
2. L_D
3. { $w \in \{$a, b$\}^*$ | w is ***not*** a palindrome }
4. { $wxyxz$ | $w, x, y, z \in \{$a, b$\}^*$ and $|x| = 5$ }
5. { $w \in \{$a, b, c, d, r$\}^*$ | w is ***not*** a substring of abracadabra }
6. { $w \in \{$a, b$\}^*$ | there is a TM M where M loops on w }
7. { $\langle M \rangle$ | M is a TM that accepts $\langle 137 \rangle$ and rejects $\langle 42 \rangle$ }
8. { $\langle M \rangle$ | M is a TM that does not accept $\langle 137 \rangle$ or does not reject $\langle 42 \rangle$ }
9. { $\langle M, n \rangle$ | M is a TM, $n \in \mathbb{N}$, and M accepts at least one string of length n }